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Market Impact and Scalability

1. Introduction

Equity systematic investment strategies using risk factor methodology has been widely in favor of international institutional investors for more than 2 decades. However, despite consistent superior results, this is an investment style that is still almost unknown to the financial industry in Brazil.

On the other hand, at Constância Investimentos systematic risk factor models are the base of our equity strategy. The model we use to build our diversified equity portfolios is very similar to the risk factor models used by major players such a as AQR, Rubeco, etc. It uses Value indicators (Price/Book, EV/EBIT, etc), Quality indicators (ROE, ROIC, etc), Growth indicators, among others.

Stocks in the IBOVESPA index are all Large Cap and liquid. However, we work with a much broader index imposing very low minimal liquidity and availability of financial information. We construct and name this index the Constância Universe. It is an equal weighted index, composed of more than 180 shares and rebalanced monthly. The Constância Universe is composed of approximately 25% small, 50% Mid and 25% Large Caps.

The systematic portfolio constructed from our Risk Factor model is very diversified with the "best stocks" having weights between 3%and 4%. Back testes of our model show an outstanding outperformance over IBOVESPA.

However, because our Risk Factor portfolio has similar composition (in terms of Large/Mid/Small Caps) to the Constância Universe, and Small/Mid Cap stocks have higher returns than Large Caps in our simulations, there are important concerns about the liquidity and scalability of implementing our systematic model because of the potential market impact of our buy/sell orders.

The cost of market impact is associated with the price variation that our buy or sale orders have on the average price that the transaction is actually executed, and is particularly important for low liquidity stocks, which is the case for most Small Cap and some Mid Cap stocks .

In this article we develop a methodology that balances the cost associated with the market impact and the attractiveness of the lower liquidity shares. Taking this cost / benefit trade-off into consideration is at the heart of our scalability analysis for our systematic strategy, and becomes increasingly important as we the asset under management (AUM) of our fund grows.

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2. Review of Market Price Impact Literature and Models

By far, the most used model of market price impact is BARRA's "square-root model" (see reference [1]).

In this model, a buy order of size Q, for a stock that has daily volume V and daily volatility σ, is assumed to have market price impact of price Δ (Q) given by the formula:

$\Delta(Q) = \alpha \left(\frac{Q}{V}\right)^{\delta} \sigma$, where α and δ are constants estimated by regressions or MLE.

Using different databases composed of hundreds of thousands to millions of trades in the US equity market, several published papers (see [1] and [3] through [8]) have estimated α to be close to 1 and δ approximately ½.

The transaction cost is related to the average execution prices, and it is usually assumed to be half of the Market Price impact. That is:

$$\text{Transaction Cost Due to Market Price Impact} = \frac{1}{2}\sqrt{\frac{Q}{V}}\sigma .$$

The Brazilian equity market lacks similar rich databases, so we will assume that the same estimates hold for the Brazilian market.

2.1. Temporary and Permant Market Impact

In a more elaborate work (see [2]), the authors use data from 700,000 trades and decompose the market impact into a permanent and a temporary component.

In this work the authors estimate that the permanent component has a power δ close to 1, however, but the size of the trades is relatively small with the order of magnitude of $\frac{Q}{V}$, on average of the order of 10-4 and 10-3.

More importantly, this paper shows that the temporary market impact component has a similar square root form with main variable $\sqrt{\frac{Q}{V.T}}\,\sigma$, where T is the execution time measure in fraction of 1 day. This implies that trades with very concentrated executions suffer a very high temporary market impact and transaction cost. Therefore, it is imperative to break down large trades into a series of smaller ones.

When we implement our systematic risk factor model, we limit the speed of execution to be 10% of 1 day volume, that is $\frac{Q}{V.T} = 0.1$. For stocks with σ= 3% (daily volatility), the cost associated with the temporary market price impact will be approximately 5 bps. This is a very modest level of transaction cost and will never be a deal breaker when considering whether or not is worth executing a trade.

Thefore, we can focus on the Permanent Market Impact. Because our model often suggests exposures equivalent to several daily volumes, the cost of permanent market impact is much more relevant and can be several tens or even hundreds of bps.

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2.2. The "Square-Root" Model is Conservative

In one of the most recent econometric work (see [9]), which uses one of the largest databases with approximately 5 million trades, the authors conclude the following:

i) For relatively small trades (smaller than 0.2% of the daily volume) the Market Impact is approximately linear.

ii) For larger trades (up to 5% of the daily volume) the square-root model is a very good fit.

iii) For large trades (close to 1 daily volume) the market price impact is even more concave, and in this range the functional form can be described with δ substantially less than ½ or even as logarithm.

All the papers cited above work with intraday data and, hence, only estimate market price impact for order of magnitude smaller than 10^0 for the variable Q/V.

In this present paper, the applied work to the Brazilian equity market, we would like to simulate the accumulation of large positions over several days, totaling several daily volumes. Therefore, we would like to extrapolate the results we have in the literature at least 1 order of magnitude above 10^0, that is to exposures of 10^1 daily volumes or more.

Therefore, we believe it is conservative to use the "square root" model to estimate our permanent market impact cost. We will keep the execution speed at a maximum of 10% of the daily volume to limit the cost associated with the temporary impact.

3. Estimating the Maket Price Impact.

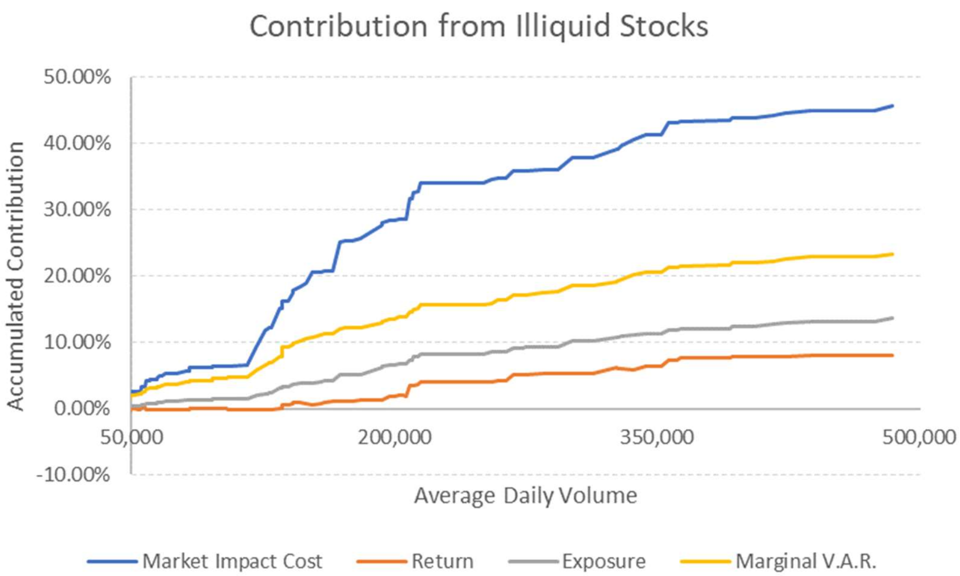
Assuming the square-root model, the table below describes the statistics from the back-test period 2004-2018 (excluding market price impact costs) for returns, volatility and market impact costs, for various levels of AUM.

AUM (R$ mm)	Market Impact Cost (% of AUM per year)	Avrg Return Model	Avrg Return IBOVESPA	Model Stdev
10	1.2%	25.2%	11.3%	21.4%
50	2.7%	25.2%	11.3%	21.4%
100	3.8%	25.2%	11.3%	21.4%
300	6.6%	25.2%	11.3%	21.4%
500	8.5%	25.2%	11.3%	21.4%
1000	12.0%	25.2%	11.3%	21.4%
2000	16.9%	25.2%	11.3%	21.4%
3000	20.7%	25.2%	11.3%	21.4%
10000	37.9%	25.2%	11.3%	21.4%

The risk and return statistics reported on the left are not affected by the AUM level but the market impact cost does depend on the level of AUM. The reason must be clear, with the increase of AUM, for a 1% exposure of the AUM in a stock, the value of Q / V will increase linearly with AUM and therefore the market impact cost will increase at the root-square rate.

Our original systematic model has an average annual simulated return of 25.2%, versus 11.3% for the IBOV. The table indicates that for an AUM of R $ 1 Bi, the 13.9% return above the IBOV of the original model would be almost all consumed by the annual cost of market price impact, which is approximately 12.0%.

In general, the most illiquid stocks are the ones that suffer the highest market price impact. The plot below shows some statistics from our back-test simulations.



Stocks with an average daily volume of up to R $ 500,000 account for:

(i) Less than 10% of the total return,

(ii) Just over 10% of the total exposure,

(iii) Over 20% of the V.A.R. (Value at risk).

(iv) Over 40% of the total market price impact

The chart strongly suggests that limiting portfolio exposure to illiquid stocks tends to benefit the balance of cost, return, risk, and scalability.

However, we don´t want to outright prohibit the inclusion of illiquid or small caps stocks in the portfolio. A stock that ranks exceptionally well in our multifactorial systematic model may merit significant allocation in a portfolio with low AUM even if it is illiquid. As we increase the AUM of the fund, it is reasonable that this illiquid exposure is diluted and cannot maintain the allocation percentage suggested by our model, due to the increase in the cost associated with the market price impact.

In the next section we describe the economic and financial rational of the methodology we use at Constância Investimentos to balance the cost of market impact and the "attractiveness" of a certain stock.

Basically, every stock, no matter how attractive it is, has a maximum number of daily volumes that is rational to have in the portfolio. If the AUM is high and the model is prescribing a larger number of daily volumes for that stock, it should be rationed.

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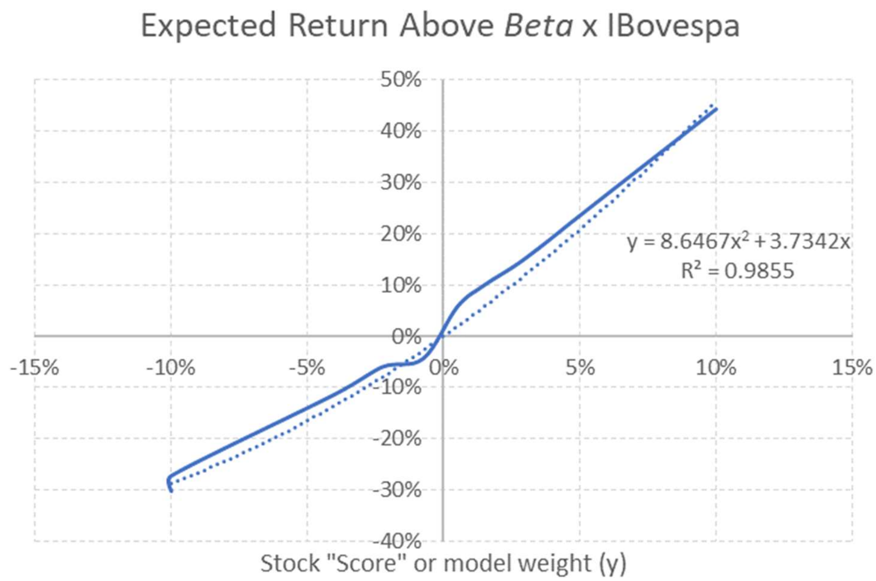
4. **Balancing Attractiveness and the Cost of Market Price Impact.**

Our systematic multifactorial model assigns a weight (or score) to each stock and generates an extremely diversified portfolio. In general, stocks considered "very good" have weights between 2% and 3% and very seldom above 4%.

4.1 Stock "Attractiveness"

If our risk factor model is good at generating profitable porfolios, then it is intuitive that the higher the score assigned to a certain stock, the more "attractive" it should be. The plot below, constructed from calculations of simple range averages, indeed suggests a monotonic relationship between the score and its expected return.



In the econometric appendix (section 7.0) we formalize this relation. Using a 2-stage FGLS we estimate that the expected annualized return of an action above the usual component *Beta* x IBOVESPA is approximately $\mu = 4$ times its score. For example, a stock that receives weight y = 3% of our systematic model and has beta equal to 1 with respect to IBOVESPA, would have an expected return approximately 12% in excess of the return explained by the general market.

Let's focus for a moment on a stock with high score and attractiveness. If we were willing to assume that the high score of this stock would perpetual, then even if the stock was illiquid and there was a huge market impact, it would always be advantageous to gradually build a relevant exposure on this stock. This conclusion follows because, once built, the benefit of the high expected return would capitalize eternally, and the initial high cost of market impact would eventually be irrelevant.

Of course, taking this perpetuity of high score and expected return of a stock for granted is totally unintuitive and unrealistic. The simulations obtained from our almost 15 years of back-testing suggest that there is a natural decay of the of the score of exceptionally attractive stocks towards zero.

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Let q = reversion speed of the score of a stock.

Using all the scores from our back test simulation (monthly data for 15 years of back-test and almost 200 stocks), we regress the scores of each y(t) over y(t-1), and obtain a monthly $q \cong 80\%$ (half-life of approximately 3 months), implying a daily $q \cong 0.99$.

This decay generates an urgency to build exposures in stocks considered very good or attractive by our multifactor model. However, as we mentioned in the previous section, in order to limit the temporary market impact, we will limit the construction speed of exposures to 10% of the volume.

Below we develop an economic and financial rationale that balances these opposing forces and takes into account attractiveness, decay, speed of execution and potential market impact cost. This methodology computes an upper limit on the maximum exposure (in terms of number of daily volumes) that a stock should have. If our multifactor model suggests an allocation above that limit, it is optimal to ration the exposure and re-shift it towards more liquid stocks.

4.2 Maximum Rational Exposure for an Illiquid Stock.

Every day, we construct the position with α = 10% of the daily volume for n days. In the Mathematical Appendix (section 6.0), we show with simple Algebra that when we use the root-square model for market impact, the number of daily volumes that maximize the stand alone cost-benefit of holding the stock is:

$$\text{Optimal maximum number of daily volumes} = \left(\frac{2}{3}.\frac{\mu/252}{1-q}\right)^2 \left(\frac{y}{\sigma}\right)^2 = k\left(\frac{y}{\sigma}\right)^2,$$

Using $q = 0.99$ and $\mu = 4$ we have $k \cong 1$.

Recalling that σ is daily volatility and therefore has values typically in the range of 2% to 3%. An "attractive" stock typically has a score y between 2% and 3% as well. In this case, the formula suggests the construction of exposures up to an upper limit of approximately 1 daily volume, which will take 10 working days because we limit the speed of execution at 10% of the daily volume.

However, it is possible to have exceptional stocks with score y = 4% and vol σ = 1%. In this case, the formula suggests maximum exposures of up to 16 daily volumes, which would take 160 working days (approximately 8 months).

The optimal number of daily volumes gives us the maximum percentage of AUM that we will be willing to invest in a stock i that has daily volume V_i:

$$\text{Maximum percentage rational allocation in stock i: } \left(\frac{y_i}{\sigma_i}\right)^2 \frac{V_i}{AUM}$$

As AUM grows, stocks invariably get diluted and exposure is re-shifted towards more liquid stocks.

When we implement of model, to maintain diversification, we limit the exposure on these more liquid stocks at 5% of AUM. To easy the notation we will ignore this fact on the next section but will come back to it later.

4.3 Rationing Illiquid Stocks and Reshifting Expoure Towards Liquid Ones.

The percentage allocation suggested by our systematic model is denoted by y. However when the stocks i has a score y_i, daily volume V_i and volatility σ_i that satisfies $\left(\frac{y_i}{\sigma_i}\right)^2 \frac{V_i}{AUM} \leq y_i$, this percentage allocation to stock should be rationed.

When AUM is increased to high levels, we need to ration the allocations of a large number of stocks, and these exposures need to be reallocated to more liquid stocks. We do this reallocation keeping the proportional exposures of the remaining stocks as long as we can, but also continuously checking if the stocks that received the extra exposure don´t violate their own upper limit exposures. This will require a recurrent process of rationing and re-shift of exposures that we detail below.

For each stock, there is a λ_{max} level which represents the highest amount of exposure increase that a stock can absorb without exceeding the optimal number of daily volumes:

$$\left(\frac{y_i}{\sigma_i}\right)^2 \frac{V_i}{AUM} = \lambda^i_{max} y_i \qquad -> \qquad \lambda^i_{max} = \frac{y_i V_i}{(\sigma_i)^2} \frac{1}{AUM}$$

We order the stocks such that: $\frac{y_1 V_1}{(\sigma_1)^2} \leq \frac{y_2 V_2}{(\sigma_2)^2} \leq \frac{y_3 V_3}{(\sigma_3)^2} \leq \cdots.$

For each ordered stock i, there exist a critical level AUM_i ($AUM_1 \leq AUM_2 \leq AUM_3 \leq \cdots$) above which the stock starts to be rationed. The first stocks to be rationed tend to be the ones with low daily volume V_i and low score y_i.

Therefore, above the cut level AUM_1 an increase in the AUM of our fund has no impact on the R$ allocation assigned to stock 1, and it stats to be diluted in percentage terms. Similar reasoning applies for each level AUM_i, above which increases in AUM of the fund would no longer impact the R$ allocation in stock i, and it is progressively diluted in terms of percentage allocation.

In the appendix, we detail the recurrent process that allows us to calculate cut-off-levels AUM_i and the proportional increases in exposure λ_i.

For extremely high levels of AUM, this process of reallocating illiquid stocks to more liquid stocks may induce the portfolio to have an excessive concentration in a few stocks. To avoid this risk, we will limit the concentration of any share to 5% of the AUM of the fund.

With these values calculated, we report in the next section the simulated results from the market impact "adjusted" model according to these process for rationing and relocation of exposures.

Because we gradually shift exposure from illiquid to more liquid stocks as AUM growth, the market price impact is reduced to fraction of the original value when AUM achieves high levels.

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5.0 Simulated Results from the Market Impact "Adjusted" Model

The table below reports the exposures of the adjusted model for the month of October 2018 and details the impacts of this re-shifting of exposures away from illiquid stocks towards more liquid stocks.

							y Ajusted For Diffeten Levels of AUM						
		AUM Cut-Level	λ Max	y Original	Volume	Volatility	50 mm	100 mm	300 mm	500 mm	1 Bi	2 Bi	5 Bi
Lowest AUM Cut-Levels	'PPLA11'	80,105	1.0000	0.02%	122,928	1.85%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
	'ATOM3'	303,068	1.0002	0.17%	110,528	2.50%	0.001%	0.001%	0.000%	0.000%	0.000%	0.000%	0.000%
	'OGXP3'	303,132	1.0002	0.05%	146,718	1.50%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
	'BPHA3'	352,522	1.0005	0.06%	325,050	2.42%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
	'INEP4'	400,582	1.0008	0.19%	178,873	2.92%	0.002%	0.001%	0.000%	0.000%	0.000%	0.000%	0.000%
	'EMAE4'	543,733	1.0019	0.58%	25,758	1.65%	0.006%	0.003%	0.001%	0.001%	0.000%	0.000%	0.000%
	'BOBR4'	549,323	1.0020	0.47%	43,933	1.93%	0.005%	0.003%	0.001%	0.001%	0.000%	0.000%	0.000%
	'GSHP3'	670,067	1.0044	0.34%	70,037	1.87%	0.005%	0.002%	0.001%	0.000%	0.000%	0.000%	0.000%
	'CTNM4'	738,422	1.0058	0.70%	35,432	1.82%	0.010%	0.005%	0.002%	0.001%	0.001%	0.000%	0.000%
	'IGBR3'	914,704	1.0098	0.36%	733,913	5.33%	0.007%	0.003%	0.001%	0.001%	0.000%	0.000%	0.000%
Largeest AUM Cut-Levels	'TRPL4'	4,383,747,352	3.3408	2.61%	32,277,989	0.76%	2.98%	3.09%	3.58%	3.80%	4.66%	5.55%	3.82%
	'CMIG4'	4,436,423,602	3.4017	0.96%	300,761,133	1.39%	1.10%	1.14%	1.32%	1.41%	1.72%	2.05%	2.91%
	'ITSA4'	4,656,192,171	3.6761	0.67%	332,826,741	1.14%	0.77%	0.80%	0.92%	0.98%	1.20%	1.43%	2.31%
	'EGIE3'	5,667,161,393	4.7732	2.26%	47,720,804	0.63%	2.58%	2.67%	3.10%	3.29%	4.03%	4.80%	8.30%
	'B3SA3'	5,710,711,329	4.8331	1.11%	344,435,767	1.18%	1.27%	1.32%	1.53%	1.62%	1.99%	2.36%	4.09%
	'BBDC4'	5,754,169,361	4.9075	0.53%	650,053,896	1.10%	0.60%	0.63%	0.72%	0.77%	0.94%	1.12%	1.94%
	'VIVT4'	5,775,149,507	4.9475	1.82%	91,089,962	0.76%	2.07%	2.15%	2.49%	2.65%	3.24%	3.86%	6.67%
	'TAEE11'	6,014,348,259	5.6416	3.17%	60,720,145	0.75%	3.62%	3.76%	4.35%	4.62%	5.67%	6.74%	11.66%
	'ITUB4'	6,257,203,621	8.4237	0.66%	893,467,514	1.06%	0.75%	0.78%	0.90%	0.96%	1.18%	1.40%	2.42%
	'VALE3'	6,305,951,827	9.5753	0.64%	1,155,742,928	1.10%	0.73%	0.75%	0.87%	0.93%	1.14%	1.35%	2.34%
Most Relevant Exposures	'AGRO3'	550,189,855	1.4880	3.99%	701,861	0.59%	4.56%	4.73%	5.48%	5.82%	3.27%	1.63%	0.65%
	'TRIS3'	140,144,683	1.2399	3.45%	613,288	1.10%	3.94%	4.09%	2.00%	1.20%	0.60%	0.30%	0.12%
	'TAEE11'	6,014,348,259	5.6416	3.17%	60,720,145	0.75%	3.62%	3.76%	4.35%	4.62%	5.67%	6.74%	11.66%
	'ALUP11'	3,612,416,734	2.7857	2.86%	13,512,720	0.62%	3.27%	3.39%	3.93%	4.17%	5.12%	6.09%	5.76%
	'TRPL4'	4,383,747,352	3.3408	2.61%	32,277,989	0.76%	2.98%	3.09%	3.58%	3.80%	4.66%	5.55%	7.64%
	'SSBR3'	111,632,276	1.2141	2.48%	447,336	0.90%	2.83%	2.93%	1.12%	0.67%	0.34%	0.17%	0.07%
	'ENEV3'	615,373,123	1.5251	2.37%	3,405,396	0.93%	2.71%	2.81%	3.26%	3.46%	2.23%	1.11%	0.45%
	'CLSC4'	63,678,083	1.1589	2.32%	391,837	1.11%	2.65%	1.71%	0.57%	0.34%	0.17%	0.09%	0.03%
	'EGIE3'	5,667,161,393	4.7732	2.26%	47,720,804	0.63%	2.58%	2.67%	3.10%	3.29%	4.03%	4.80%	8.30%
	'SULA11'	2,609,623,809	2.3806	2.25%	31,170,940	1.06%	2.56%	2.66%	3.08%	3.27%	4.01%	4.77%	2.79%

The stocks in the upper block of the table have low original scores and low daily average volumes. As a result, these stocks have low AUM cutoff levels, small exposure increases (low max lambda) before violating the maximum daily volume number $(y_i/\sigma_i)^2$ and are the first to be diluted when it increases the bottom AUM (see columns to the right).

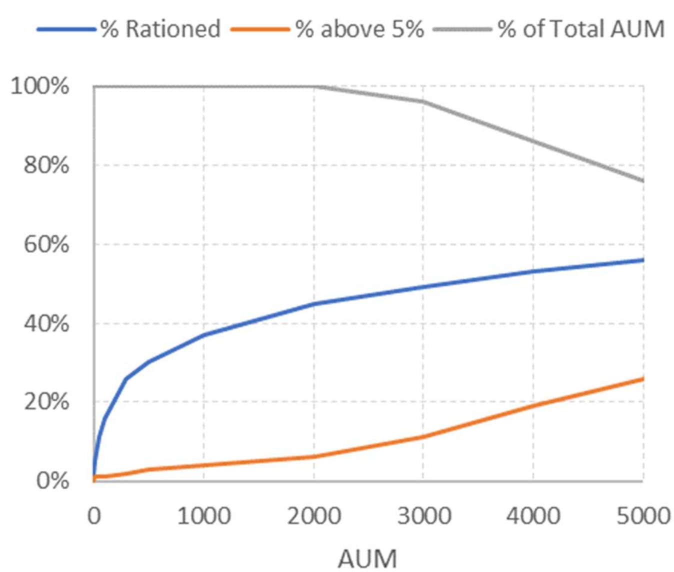
At the other extreme of the spectrum, the stocks in the middle block of the table have high original scores and/or have a very high average daily volume. As a result, these socks have high AUM cutoff levels and high capacity to absorb exposure increases (maximum high lambda) caused by the eventual rationing of less liquid stocks.

For example, ITUB4 has a relatively modest 0.66% initial score in the original model. However, because of the high liquidity of this stock, this percentage exposure can be increased up to 8.4 x 0.66% = 5.6% without the maximum number of daily volumes being violated.

TAEE11 is another example of stock with high capacity to absorb the exposures that are rationed of shares less liquid as the AUM is increased. It can be located both in the second and in the third block of stocks (most relevant exposures) in the table above. This stock doesn´t rank among the most liquid ones. However, it´s high original score y = 3.17% and low daily σ = 0.75% allows this stocks to have very high maximum ration daily volumes $\left(\frac{y}{\sigma}\right)^2 \cong 18$. This is reflected in λ_{max} = 5.64, and in the capacity to have the original exposure increased up to 5.64 x 3.17% = 17.9%.

Even though this scenario would only occur for very high levels of AUM, this magnitude of concentration seems excessive. To maintain minimum diversification and controlled concentration risk, we will limit the percentage allocated to any share to a maximum of 5%. Allocations above 5% are redistributed proportionately to other actions, but always respecting the limit imposed by the maximum number of daily volumes.

The plot below shows how the following measures evolve as we increase the AUM: (i) the percentage of AUM rationed because of the maximum number of daily volumes, (ii) the percentage of AUM that we need to force below 5% exposure and (iii) the total exposure



In the above statistics suggest that for AUM values around R $ 3 Bi, it is no longer possible to do this re-allocation because all stocks have reached a maximum of 5% or the maximum number of daily volumes, and the total exposure falls below of 100%.

This AUM threshold value defines the current maximum capacity of the strategy. Above this value, it does not make sense to allocate more resources to this systematic strategy. Naturally this maximum capacity improves as the market becomes more liquid and daily volumes of stocks increase.

Implementing the above redemptions from less liquid shares to more liquid stocks substantially reduces the cost of market impact and more than compensates for the reduction in expected return as we reallocate from Small Caps shares (usually less liquid) to Mid and Large Caps shares (in general more liquid).

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Below we have the following statistics tables for the backtest period from August 2004 to October 2018:

AUM	Market Impact Cost (% of AUM / year)		Avrg Return			Stdev		Market Cap		
(R$ mm)	Original	Adjusted	Original	Adjusted	IBOVESPA	Original	Adjusted	Small	Mid	Large
10	1.2%	0.8%	25.2%	25.1%	11.3%	21.4%	21.2%	29%	49%	22%
50	2.7%	1.5%	25.2%	24.3%	11.3%	21.4%	20.3%	23%	52%	25%
100	3.8%	1.8%	25.2%	23.7%	11.3%	21.4%	20.0%	19%	55%	26%
300	6.6%	2.5%	25.2%	22.7%	11.3%	21.4%	19.8%	11%	56%	33%
500	8.5%	2.9%	25.2%	22.2%	11.3%	21.4%	19.9%	7%	55%	38%
1000	12.0%	3.6%	25.2%	22.0%	11.3%	21.4%	20.6%	5%	49%	46%
2000	16.9%	4.3%	25.2%	21.6%	11.3%	21.4%	21.2%	4%	39%	57%
3000	20.7%	4.8%	25.2%	20.8%	11.3%	21.4%	21.2%	3%	34%	63%
10000	37.9%	5.9%	25.2%	18.7%	11.3%	21.4%	21.5%	3%	29%	68%

As we have already mentioned, the original model has an average annual back-test return of 25.2%, versus 11.3% for the IBOV. The table indicates that for an AUM of R $ 1 Bi, the 13.9% excess return (to the IBOV) of the original model would be almost all consumed by the annual Cost of Market Impact, which is approximately 12.0%.

However, the modified model suggests in the back-test an over performance of 22% - 11.3% = 10.4% and a Market Impact cost of only 3.6%. This "cost savings" of Market Impact is obtained because, as we increase the AUM to R $ 1b, the modified model reduces exposure to Small Caps (from 30% to 5%) and increases exposure to Mid Caps and Large Caps. Also note that reducing the exposure to small caps is very similar to the reduction to exposures of illiquid stocks (Volume <1mm / day).

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6.0 Mathematical Appendix.

To obtain the optimal number of daily exposure volumes in a attractive and illiquid action, let us assume that every day we construct the position at the maximum velocity of $\alpha = 10\%$ of the daily volume for n days. We will have the following expected daily returns:

Day 1: $r_1 = \frac{\mu y}{252} * \alpha.V$

Day 2: $r_2 = \frac{\mu y q}{252} * 2\alpha.V = 2q.r_1$

Day 3: $r_3 = \frac{\mu y q^2}{252} * 3\alpha.V = 3q^2.r_1$

.....

Day n: $r_n = \frac{\mu y q^{n-1}}{252} * \alpha.V = nq^{n-1}.r_1$

Etc,

The sum of these expected returns is: $\frac{1-q^n}{(1-q)^2} r_1 = \frac{1-q^n}{(1-q)^2} \left(\frac{\mu y}{252} * \alpha.V\right)$

The market impact cost of building and eventually unwinding this position estimated by the square-root model is: $\alpha V n \sqrt{\frac{\alpha V n}{V}} \sigma = V(\alpha n)^{3/2} \sigma$

The expected return net of costs is approximately: $\frac{1-q^n}{(1-q)^2} \left(\frac{\mu y}{252} * \alpha.V\right) - V(\alpha n)^{3/2} \sigma$

We will make the first order approximation $1 - q^n \approx n(1-q)$. With $q = 0.99$, this approximation is very reasonable up to n =60 (3 months). The expected return net of costs becomes:

$$\frac{n(1-q)}{(1-q)^2} \left(\frac{\mu y}{252} * \alpha.V\right) - V(\alpha n)^{3/2} \sigma = V\left[\frac{1}{1-q} * \frac{\mu y}{252} * \alpha n . -(\alpha n)^{3/2} \sigma\right]$$

We take first order conditions with respect to the number of daily volumes $z = \alpha n$

$$\frac{1}{1-q} * \frac{\mu y}{252} - \frac{3}{2} (\alpha n^*)^{1/2} \sigma = 0$$

Optimal maximum number of daily volumes = $(\alpha n)^* = \left(\frac{2}{3} . \frac{\mu/252}{1-q}\right)^2 \left(\frac{y}{\sigma}\right)^2 = k \left(\frac{y}{\sigma}\right)^2$

,

Using $q = 0.99$ and $\mu = 4$ we have $k \cong 1$.

This implies:

Maximum percentage rational allocation in stock i: $\left(\frac{y_i}{\sigma_i}\right)^2 \frac{V_i}{AUM}$

The percentage of allocation suggested by our systematic model is y. However when the

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stocks i has a score y_i, daily volume V_i and volatility σ_i such that $\left(\frac{y_i}{\sigma_i}\right)^2 \frac{V_i}{AUM} \leq y_i$, the percentage allocation to stock is rationed.

As we increase the AUM, we need to ration the allocations of an increasing number of stocks, and these exposures need to be reallocated to more liquid stocks. We need to do this reallocation keeping the proportional exposures of the remaining stocks as long as we can, but also continuously checking if the stocks that received the extra exposure don´t violate their upper limit exposures. This will require a recurrent re-shift that we explain below.

For each relatively liquid share, there is a λ_{max} which represents the highest amount of exposure increase that an action can absorb without exceeding the optimal number of daily volumes:

$$\left(\frac{y_i}{\sigma_i}\right)^2 \frac{V_i}{AUM} = \lambda_{max}^i y_i \quad -> \quad \lambda_{max}^i = \frac{y_i V_i}{(\sigma_i)^2} \frac{1}{\text{AUM}}$$

We order the stocks such that: $\frac{y_1 V_1}{(\sigma_1)^2} \leq \frac{y_2 V_2}{(\sigma_2)^2} \leq \frac{y_3 V_3}{(\sigma_3)^2} \leq \cdots.$

The variables AUM_i and λ_i need to be computed recurrently:

<u>Step 1</u>: If $\text{AUM} \leq \text{AUM}_1$ then every stock receives allocation according to its original score y_i. The are no rationings or re-shifting of exposures.

By construction we have $y_1 + y_2 + .. + y_N = 1$. In this case, λ_1=1 and we compute AUM_1 as the solution of:

$$\left(\frac{y_1}{\sigma_1}\right)^2 \frac{V_1}{\text{AUM}_1} = \lambda_1 y_1 \quad \Rightarrow \quad \lambda_1 \text{AUM}_1 = \frac{y_1 V_1}{(\sigma_1)^2}$$

<u>Step 2</u>: If $\text{AUM}_1 < \text{AUM} \leq \text{AUM}_2$ then stock 1 receives the reduced allocation $\left(\frac{y_i}{\sigma_i}\right)^2 \frac{V_i}{\text{AUM}} = \lambda_1 y_1 \frac{\text{AUM}_1}{\text{AUM}}$, and the exposure to the other stocks will have a proportional increase $\lambda_2 > 1$ computed from:

$$y_1 \frac{\text{AUM}_1}{\text{AUM}} + \lambda_2 (y_2 + y_3 + .. + y_N) = 1 \quad \text{or}$$

$$y_1 \frac{\text{AUM}_1}{\text{AUM}} + \lambda_2 (1 - y_1) = 1$$

As we increase AUM, this increase in expoures by the factor λ_2 is only feasible up to the level AUM = AUM_2 when stock 2 reaches its maximum allowed R$ allocation.

$$\left(\frac{y_2}{\sigma_2}\right)^2 \frac{V_2}{\text{AUM}_2} = \lambda_2 y_2 \quad \Rightarrow \quad \lambda_2 = \frac{y_2}{(\sigma_2)^2} \frac{V_2}{\text{AUM}_2}$$

Substituting in the previous equation we can compute AUM_2:

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$$y_1 \frac{AUM_1}{AUM_2} + \frac{y_2}{(\sigma_2)^2} \frac{V_2}{AUM_2} (1 - y_1) = 1$$

$$AUM_2 = \lambda_1 y_1 AUM_1 + (1 - y_1) \frac{y_2 V_2}{(\sigma_2)^2}$$

<u>Step 3</u>: If $AUM_2 < AUM \leq AUM_3$ the stock 1 receives reduced exposure $y_1 \frac{AUM_1}{AUM}$, stock 2 receives reduced exposure $\lambda_2 y_2 \frac{AUM_2}{AUM}$ and the other stocks receive increase in the original exposure by the factor $\lambda_3 > \lambda_2$ which satisfies:

$$\lambda_1 y_1 \frac{AUM_1}{AUM} + \lambda_2 y_2 \frac{AUM_2}{AUM} + \lambda_3 (y_3 + y_4 + .. + y_N) = 1 \quad \text{or}$$

$$y_1 \frac{AUM_1}{AUM} + \lambda_1 y_2 \frac{AUM_2}{AUM} + \lambda_3 (1 - y_1 - y_2) = 1$$

This proportional increase by the factor λ_2 can only hold up-until $AUM = AUM_3$, when stock 3 reached is maximum capacity.

$$\left(\frac{y_3}{\sigma_3}\right)^2 \frac{V_3}{AUM_3} = \lambda_3 y_3 \quad \text{or } \lambda_3 = \frac{y_3}{(\sigma_3)^2} \frac{V_3}{AUM_3}$$

Substituting in the previous equation we can compute AUM_3:

$$\lambda_1 y_1 \frac{AUM_1}{AUM_3} + \lambda_2 y_2 \frac{AUM_2}{AUM_3} + 6 \frac{y_3}{(\sigma_3)^2} \frac{V_3}{AUM_3} (1 - y_1 - y_2) = 1 \quad \text{ou}$$

$$AUM_3 = \lambda_1 y_1 AUM_1 + \lambda_2 y_2 AUM_2 + (1 - y_1 - y_2) \frac{y_3 V_3}{(\sigma_3)^2}$$

<u>Generic step j+1</u>: If $AUM_j < AUM \leq AUM_{j+1}$ then every stock *i* with $i \leq j$ receives reduced allocation $\lambda_i y_i \frac{AUM_i}{AUM}$ and the remaining stocks increased exposure by the factor λ_{j+1} up-until $AUM = AUM_{j+1}$ calculated by the following recurrence:

$$AUM_{j+1} = \lambda_1 y_1 AUM_1 + \lambda_2 y_2 AUM_2 + \cdots + \lambda_j y_j AUM_j + \left(1 - y_1 - y_2 - \cdots - y_j\right) \frac{y_{j+1} V_{j+1}}{\left(\sigma_{j+1}\right)^2}$$

$$\lambda_{j+1} = \frac{y_{j+1}}{\left(\sigma_{j+1}\right)^2} \frac{V_{j+1}}{AUM_{j+1}}$$

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7.0 Econometric Appendix.

Let $x_{j,t}$ be the score of stock j at time t, and $y_{j,t}$ the return over o stock j at time t minus retun of Ibovespa x BetaIbov, where BetaIbov is computed with the usual formula Covar(ReturnsStock,ReturnsIBOV)/VAR(IBOV) using the full time series data up to time t.

We will assume the following fuction form holds.

$$y_{j,t} = \beta_0 + \beta_1 x_{j,t} + \beta_2 (x_{j,t})^2 + \varepsilon_{j,t} \qquad VAR(\varepsilon_{j,t}) = \sigma_{j,t}^2$$
$$\sigma_{j,t} = \gamma_{0,j} + \gamma_1 x_{j,t} + \gamma_2 (x_{j,t})^2$$

The goal is to estimate the beta and gamma parameters by regression and to test if they are significantly different from zero. We have 175 months and approximately 190 actions to estimate these parameters.

Since volatilities are unknown and assumed to be time-varing (heteroscedasticity) and there are correlations between de stocks, we will apply 2-stage FGLS (instead of OLS).

The estimators and confidence intervals of the FGLS regressions are obtained by inversions and matrix multiplications using MatLab.

To set up the regression in matrix notion we first piled up de stocks:

$$y_t = X_t.\beta + \varepsilon_t \qquad \text{where} \quad X_t = [1 \;\; x_t \;\; (x_t)^2] \quad , \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} \quad \text{and}$$

$$VAR(\varepsilon^t) = \Omega^{t,atvs} = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \dots & \rho_{1J}\sigma_1\sigma_J \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \dots & \rho_{2J}\sigma_2\sigma_J \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1J}\sigma_1\sigma_J & \rho_{2J}\sigma_2\sigma_J & \dots & \sigma_J^2 \end{bmatrix}$$

where x_t and y_t are vectors with the same size as the number of stocks, which is 190.

Next we pile up all the 175 time periods (months) to obtain:

$$y = X.\beta + \varepsilon \qquad \text{where} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_T \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_T \end{bmatrix} \quad \text{have are (175 x 190) by 1 vectors,}$$

$$VAR(\varepsilon) = \Omega = \begin{bmatrix} \Omega^{1,atvs} & 0 & \dots & 0 \\ 0 & \Omega^{2,atvs} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Omega^{T,atvs} \end{bmatrix}$$

(The zeros are, in fact, 175x175 matrices and indicate time series correlation of zero)

When Ω was known, the most effcient linear estimator $\hat{\beta}_T^{GLS}$ is obtained by minimizing the weighted square erros $\varepsilon'.\Omega^{-1}.\varepsilon$. First order conditions give us:

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$\hat{\beta}_T^{GLS} = (X'\Omega^{-1}X)^{-1}X\Omega^{-1\prime}y$

By the Central Limt Theorem : $\sqrt{T}[\hat{\beta}_T^{GLS} - \beta] \xrightarrow{T\to\infty} N\left(0, \left(\frac{1}{T}X'\Omega^{-1}X\right)^{-1}\right)$, and the confidence intervals come from the Var-Covar matrix $\left(\frac{1}{T}X'\Omega^{-1}X\right)^{-1}$.

Hence, to test the null hypothesis $\beta = 0$, the standard econometric procedure is to calculate $\hat{\beta}_T^{GLS}$ and $\mathrm{diag}((X'\Omega^{-1}X)^{-1})$.

Since we don´t know Ω , we need to estimate it and apply FGLS with a 2 stage method. In the first stag, we ignore Ω and use OLS (as if $\Omega = \sigma^2.I$ was true)

$$\hat{\beta}_{OLS} = \begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\end{bmatrix} = (X'X)^{-1}X'y = \begin{bmatrix}-0.00014976\\ 0.332294\\ -3.195464\end{bmatrix}$$

Next we obtain the sample error $\varepsilon_{OLS} = y - X.\ \hat{\beta}_{OLS}$, calculate the estimate $\hat{\sigma} = 11.54\%$ and the matrix $(X'\Omega^{-1}X)^{-1} = \left(\frac{X'X}{\hat{\sigma}^2}\right)^{-1}$

New to MATLAB? See resources for Getting Started.

```
>> BetaOLS = inv(X'*X)*X'*y;
>> BetaOLS = inv(X'*X)*X'*y

BetaOLS =

  -0.000149763066843
   0.332293849012949
  -3.195469446145568

>> erro = y - X*BetaOLS;
>> sigma = std(erro)

sigma =

   0.115411241427047

>> Varianc = (X'*X/sigma^2)^(-1)

Varianc =

   0.000000153448179  -0.000012188222942   0.000141691633615
  -0.000012188222942   0.011294491825391  -0.189573591518627
   0.000141691633615  -0.189573591518627   4.759305207170249

>> vols = diag(Varianc).^(1/2)

vols =

   0.000391724621772
   0.106275546695329
   2.181583188230568

fx
```

The above results indicate that, if we were to use OLS, we would reject the hypothesis $\beta_1 = 0$ with approximately3 standard deviation. Note that $\beta_1 = 0.3322$ was obtained with monthly returns, so in annual terms we have 12 x $\beta_1 = 4$.

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With the functional form that we assumed for the volatility, the standard procedure is to analyze the module of the OLS errors:

$$\frac{|\varepsilon_{OLS}^{j,t}|}{c} = \gamma_{0,j} + \gamma_1 x_{j,t} + \gamma_2 (x_{j,t})^2 \ + \ \text{<noise>} \quad ,$$

where c is a constant that we compute form the sample data $\frac{|E(\varepsilon_{OLS}^{j,t})|}{\sqrt{E(\varepsilon_{OLS}^{j,t})^2}} = 0.38$

If $\gamma_{0,j}$ did not depend on j, we could simply regress the 175x190 erros over the variable [1, $x_{j,t}$ and $(x_{j,t})^2$]. Since $\gamma_{0,j}$ don´t depend on j, we remove it by subtracting out the time-average of each stock.

$$|\varepsilon_{OLS}^{j,t}| - \frac{\Sigma_t |\varepsilon_{OLS}^{j,t}|}{T} = \gamma_1 \left[x_{j,t} - \frac{\Sigma_t x_{j,t}}{T}\right] + \gamma_2 \left[(x_{j,t})^2 - \frac{\Sigma_t (x_{j,t})^2}{T}\right] + \ \text{<noise>}$$

The coefficients γ_1 and γ_2 are then obtained through the OLS regression of $\varepsilon_{OLS}^{j,t}$ over the nomalized matrix of X

```
ommand Window
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>> Xnorm=[Pesosnormstrip Pesosnormstrip2];
>> gama12 = inv(Xnorm'*Xnorm)*Xnorm'*erroabsstripnorm

gama12 =

    0.139111870077944
    7.448861173679919

>>
```

$\gamma_{0,j}$ are obtained from:

$$\frac{\Sigma_t (\varepsilon_{OLS}^{j,t})^2}{T} = \gamma_{0,j} + \gamma_1 \left[\frac{\Sigma_t x_{j,t}}{T}\right] + \gamma_2 \left[\frac{\Sigma_t (x_{j,t})^2}{T}\right]$$

We are now ready to compute an estimate for Ω:

$$\widehat{\Omega}_t^{atvs} = \begin{bmatrix} \sigma_{1,t}^2 & \rho_{12}\sigma_1\sigma_2 & \dots & \rho_{1J}\sigma_1\sigma_J \\ \rho_{12}\sigma_1\sigma_2 & \sigma_{2,t}^2 & \dots & \rho_{2J}\sigma_2\sigma_J \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1J}\sigma_1\sigma_J & \rho_{2J}\sigma_2\sigma_J & \dots & \sigma_{J,T}^2 \end{bmatrix} \quad e \quad \widehat{\Omega} = \begin{bmatrix} \widehat{\Omega}_1^{atvs} & 0 & \dots & 0 \\ 0 & \widehat{\Omega}_2^{atvs} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \widehat{\Omega}_T^{atvs} \end{bmatrix}$$

And finally apply FGLS:

$$\hat{\beta}_T^{FGLS} = (X'\widehat{\Omega}^{-1}X)^{-1} X'\widehat{\Omega}^{-1} y \quad , \quad VAR(\hat{\beta}_T^{FGLS}) = (X'\widehat{\Omega}^{-1}X)^{-1}$$

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The FGLS estimates have similar order of magnitude as the results we had obtained for OLS.

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```
>> BigMAtrixHo = kron(sparse(A),B);
>> VariancFGLS = (X_vol'*(BigMAtrixHo)*X_vol)^(-1)

VariancFGLS =

   0.000000004672840   0.000000039545147  -0.000006155406307
   0.000000039545147   0.000038003761495   0.000535514110072
  -0.000006155406307   0.000535514110072   0.062701617663835

>> volsFGLS = diag(VariancFGLS).^(1/2)

volsFGLS =

   0.000068358171692
   0.006164719092944
   0.250402910653681

>> BetaFGLS = VariancFGLS*X_vol'*BigMAtrixHo*y_vol

BetaFGLS =

   0.010835099271540
   0.412101176765303
  -0.026494032633952
```


Note how the standard deviations of $Beta_{FGLS}$ are much smaller than the stand deviations of $Beta_{OLS}$. That is, our FGLS estimators are more efficient than our OLS estimators as it should be.

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